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Non-linear stability analysis of film flow down a heated or cooled inclined plane with viscosity variation

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Abstract—Non-linear kinematic equations for film thickness which takes into account the effect of viscosity variation governed by the Arrhenius-type relation are used to investigate the non-linear stability of film flows. The results show that cooling (heating) from the wall will stabilize (destabilize) the film flows both linearly and nonlinearly. The supercritical stability and subcritical instability both prove possible here with higher heating tending to reduce the threshold amplitude in the subcritical unstable region and increase the amplitude of supercritical waves. Stability is also influenced by the Prandtl number in the way that stability increases (decreases) as the Prandtl number increases when cooling (heating).

INTRODUCTION

THE FLOW of thin liquid films has been shown to be linearly unstable with respect to surface waves [1, 2]. The non-linear modification of linear waves was studied by Benney [3], but the effect of surface tension was not included, so that the solution had no tendency towards a finite amplitude equilibrium state. If the effect of surface tension is included, then the supercritical stability was found to be possible [4–6]. Further, Anshus [4] reported that non-linear instability appeared in the region near the upper branch of the neutral stability curve, which was just opposed to ref. [6] finding that such instability existed near the lower branch of the neutral stability curve in the α–Re plane.

The effects of the temperature gradient across a film with constant viscosity and with phase change on the interface has been investigated by Unsal and Thomas [7, 8], Spinder [9] and Kocamustafaogullari [10]. However, as the gradient tends to be high, the assumption of constant viscosity can hardly be justified. Shair [11] and Yih and Seagrave [12] have also studied this problem by assuming that viscosity varies continuously with depth in a fixed manner. All their studies come to the same conclusion that heating from the wall destabilizes the film flow, while cooling stabilizes the system.

Since these studies assumed a fixed depth dependence for the viscosity, perturbation of the viscosity was not allowed, as indicated by Spindle [13]. It was Craik and Smith [14] and Goussis and Kelly [15, 16] that started considering the effect of viscosity perturbation. Craik and Smith assumed that the viscosity of a fluid element remains constant as it is convected in the flow, and so were thought to have ignored the effects of diffusion. While Goussis and Kelly revealed that, in the case of cooling, a cut-off Prandtl number, above which the flow turned to be linearly stable with respect to long waves and linearly unstable with respect to short waves [16], did exist. Apparently, most of their studies were addressed to linear theory analyses.

This paper studies the non-linear stability of liquid film flows with viscosity variation depending exponentially on temperature on a valid-for-long-waves basis. Effects of surface tension and perturbations of the viscosity are also included.

THE NON-LINEAR KINEMATIC EQUATION

Consider a liquid film flow down an inclined plane as shown in Fig. 1. With all properties being constant except viscosity that varies with temperature according to the Arrhenius-type relation, the governing equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho g\sin\gamma+\frac{\partial}{\partial x}\left[2\mu\left(\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right] \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}-\rho g\cos\gamma+\frac{\partial}{\partial x}\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\left(\frac{\partial v}{\partial y}\right)\right] \tag{3}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

where u, v, are the velocities, ρ the density, c_p the

NOMENCLATURE

A, B, C	functions of parameters
Ar	Arrhenius number
c	complex wave velocity, $c_r + ic_i$
c_p	specific heat of liquid
g	gravitational acceleration
h	film thickness
k	thermal conductivity of liquid
L, L_0, L_1	operators
p_g	pressure of gas
Pr	Prandtl number
Pr_c	cut-off Prandtl number
R	radius of curvature of the free surface
Re	Reynolds number
Re_c	critical Reynolds number
S	surface tension
t	time
t_0, t_1	time scales
T	temperature
T_w	temperature of the wall
T_s	temperature on the free surface
u	velocity in the x-direction
U	reference velocity
v	velocity in the y-direction
x, y	spatial coordinates.

Greek symbols

α	dimensionless wave number
β	parameter indicating the gradient of viscosity
γ	angle of inclination
ε	small parameter
η	perturbation of film thickness
θ	dimensionless temperature
λ	wavelength of disturbance
Λ	increment of wave speed
μ	viscosity
ν	kinematic viscosity
ρ	density
σ	dimensionless surface tension
τ_n	normal stress
τ_s	shear stress.

Subscripts

$x, y, t, \ldots$	differentiation with respect to $x, y, t, \ldots$

Superscript

$*$	dimensionless quantity.

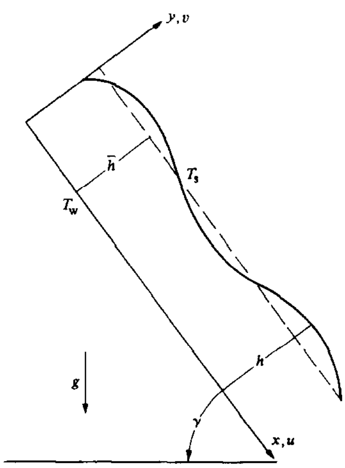

FIG. 1. Diagram of the film flow system.

specific heat, p the pressure, T the temperature and k the thermal conductivity of the liquid. The viscosity μ, which is dependent on temperature, is given as follows:

$$\frac{\mu}{\rho} = \frac{\mu_s}{\rho} \exp\left[-Ar(T-T_s)/T_s\right]$$

where T_s is the temperature at the free surface and Ar the Arrhenius number. For the boundary conditions, the no-slip and constant temperature conditions at the wall are

$$u = v = 0, \quad T = T_w \quad \text{at} \quad y = 0. \tag{5}$$

At the free surface, the constant temperature, the balance of normal and tangential forces, and the kinematic conditions are

$$T = T_s, \quad \tau_s = 0, \quad \tau_n - \frac{S}{R} = -p_g, \quad \frac{Dh}{Dt} = v \quad \text{at} \quad y = h \tag{6}$$

where τ_s, τ_n, p_g are the shear stress, the normal stress, and the gas pressure at the free surface, respectively, S the surface tension and R the radius of curvature of the free surface. After defining some dimensionless variables as follows:

$$\alpha = \frac{2\pi\bar{h}}{\lambda}, \quad \frac{p-p_g}{\rho U^2} = p^*, \quad h^* = \frac{h}{\bar{h}}, \quad x^* = \frac{\alpha x}{\bar{h}}, \quad y^* = \frac{y}{\bar{h}}$$

$$t^* = \frac{\alpha\bar{h}t}{U}, \quad u^* = \frac{u}{U}, \quad v^* = \frac{v}{\alpha U}$$

$$Pe = Re\,Pr, \quad \theta = \frac{T-T_s}{T_w-T_s}$$

$$\sigma = \left[\frac{s^3}{2^4\nu_s^4\rho g \sin\gamma}\right]^{1/3}, \quad Re = \frac{U\bar{h}}{\nu_s}, \quad \beta = \frac{-Ar}{T_s}(T_w - T_s) \tag{7}$$

where

$$U = \frac{g \sin\gamma \bar{h}^2}{\nu_s} N, \quad N = \left(\frac{1}{\beta}\right)^2 [1 - e^{-\beta}(\beta+1)]$$

the governing equations and boundary conditions become (the respective stars are dropped)

$$u_x + v_y = 0 \tag{8}$$

$$[e^{\beta\theta} u_y]_y + \frac{1}{N} = \alpha Re[p_x + u_t + uu_x + vu_y] + [2(e^{\beta\theta} u_x)_x + (e^{\beta\theta} v_x)_y] \tag{9}$$

$$p_y + \frac{2}{Re}\cot\beta = \frac{\alpha}{Re}[(e^{\beta\theta} u_y)_x + (2e^{\beta\theta} v_y)_y] + \frac{\alpha^3}{Re}(e^{\beta\theta} v_x)_x \tag{10}$$

$$\theta_{yy} = \alpha Pe(\theta_t + u\theta_x + v\theta_y) - \alpha^2\theta_{xx} \tag{11}$$

$$u = 0 = v, \quad \theta = 1 \quad \text{at} \quad y = 0 \tag{12}$$

$$\theta = 0 \quad \text{at} \quad y = h \tag{13}$$

$$(u_y + \alpha^2 v_x)(1 - \alpha^2 h_x) - 4\alpha^2 u_x h_x = 0 \quad \text{at} \quad y = h \tag{14}$$

$$p + 2\frac{\alpha}{Re} u_x (1 + \alpha^2 h_x^2)(1 - \alpha^2 h_x^2)^{-1} + 2\alpha^2 \sigma N^{-1/3} Re^{-5/3} h_{xx}(1 + \alpha^2 h_x^2)^{-3/2} = 0 \quad \text{at} \quad y = h \tag{15}$$

$$h_t + uh_x = v \quad \text{at} \quad y = h. \tag{16}$$

For the isothermal problem, the film flow becomes unstable under long wave perturbations (small wave number), then the solutions of the above system could be obtained in the following way. By introducing the expansions

$$\begin{aligned} u &= u_0 + \alpha u_1 + \cdots \\ v &= v_0 + \alpha v_1 + \cdots \\ p &= p_0 + \alpha p_1 + \cdots \\ \theta &= \theta_0 + \alpha\theta_1 + \cdots \end{aligned} \tag{17}$$

then substituting equations (17) into equations (8)–(16) and collecting terms by order which enable us to obtain the following zero- and first-order systems:

$O(\alpha^0)$

$$(e^{\beta\theta_0} u_{0y})_y + \frac{1}{N} = 0 \tag{18}$$

$$p_{0y} + \frac{2}{Re}\cot\gamma = 0 \tag{19}$$

$$\theta_{0yy} = 0 \tag{20}$$

$$u_0 = 0 = v_0, \quad \theta_0 = 1 \quad \text{at} \quad y = 0 \tag{21}$$

$$\theta_0 = 0 \quad \text{at} \quad y = h \tag{22}$$

$$u_{0y} = 0 \quad \text{at} \quad y = h \tag{23}$$

$$p_0 = -2\alpha^2 \sigma N^{-1/3} Re^{-5/3} h_{xx} \quad \text{at} \quad y = h; \tag{24}$$

$O(\alpha)$

$$[e^{\beta\theta_0}(u_{1y} + \beta u_{0y}\theta_1)]_y = Re[p_{0x} + u_{0t} + u_0 u_{0x} + v_0 u_{0y}] \tag{25}$$

$$p_{1y} = \frac{1}{Re}[(e^{\beta\theta_0} u_{0y})_x + (2e^{\beta\theta_0} v_{0x})_y] \tag{26}$$

$$\theta_{1yy} = Pe[\theta_{0t} + u_0\theta_{0x} + v_0\theta_{0y}] \tag{27}$$

$$u_1 = 0 = v_1, \quad \theta_1 = 0 \quad \text{at} \quad y = 0 \tag{28}$$

$$u_{1y} = 0 \quad \text{at} \quad y = h \tag{29}$$

$$p_1 + \frac{2}{Re} u_{0x} = 0 \quad \text{at} \quad y = h. \tag{30}$$

After a long and tedious procedure, the zero- and first-order solutions are as follows:

$$u_0 = l_1(a_1 h_y e^{(\beta/h)y} + a_2 h^2 e^{(\beta/h)y} - a_2 h^2)$$

$$v_0 = l_1 h_x (a_3 y^2 e^{(\beta/h)y} + a_4 h_y e^{(\beta/h)y} a_5 h^2 e^{(\beta/h)y} + a_6 h_y - a_5 h^2)$$

$$p_0 = \frac{2}{Re}\cot\gamma(h - y) + 2\alpha^2\sigma N^{-1/3} Re^{-5/3} h_{xx}$$

$$\theta_0 = 1 - \frac{y}{h}$$

$$\theta_1 = \frac{Pe\, h_t}{6h^2}(y^3 - h^2 y) + Pe\, L_1 h_x (d_1 h y^2 e^{(\beta/h)y} + d_2 h^2 y e^{(\beta/h)y} d_3 h^3 e^{(\beta/h)y} + d_4 y^3 + d_5 h y^2 + d_6 h^2 y - d_8 h^3)$$

$$\begin{aligned} u_1 = {} & \frac{Pe\, h_t}{6h^2}(r_1 h y^4 e^{(\beta/h)y} + r_2 h^2 y^3 e^{(\beta/h)y} \\ & + r_3 h^3 y^2 e^{(\beta/h)y} r_4 h^4 y e^{(\beta/h)y} + r_5 h^5 e^{(\beta/h)y} \\ & + r_6 h^5) + Pe\, l_1 h_x (r_7 h^2 y^3 e^{(2\beta/h)y} \\ & + r_8 h^3 y^2 e^{(2\beta/h)y} + r_9 h^4 y e^{(2\beta/h)y} + r_{10} h^5 e^{(2\beta/h)y} \\ & + r_{11} h y^4 e^{(\beta/h)y} + r_{12} h^2 y^3 e^{(\beta/h)y} \\ & + r_{13} h^3 y^2 e^{(\beta/h)y} + r_{14} h^4 y e^{(\beta/h)y} \\ & + r_{15} h^5 e^{(\beta/h)y} + r_{16} h^5) + Re\left(\frac{2}{Re}\cot\gamma h_x \right. \\ & \left. - 2\alpha^2\sigma N^{-1/3} Re^{-5/3} h_{xxx}\right)(r_{19} h y e^{(\beta/h)y} \\ & + r_{18} h^2 e^{(\beta/h)y} + r_{19} h^2) + Re\, l_1 h_x (r_{20} h y^2 e^{(2\beta/h)y} \\ & + r_{21} h^2 y e^{(2\beta/h)y} + r_{22} h^3 e^{(2\beta/h)y} \\ & + r_{23} h^2 y e^{(\beta/h)y} + r_{24} h^3 e^{(\beta/h)y} + r_{25} h^3) \\ & + Re\, l_1 h (r_{26} h^2 y^3 e^{(3\beta/h)y} + r_{27} h^3 y^2 e^{(3\beta/h)y} \\ & + r_{28} h^4 y e^{(3\beta/h)y} + r_{29} h^5 y e^{(3\beta/h)y} \\ & + r_{30} h^3 y^2 e^{(2\beta/h)y} + r_{31} h^4 y e^{(2\beta/h)y} \\ & + r_{32} h^5 e^{(2\beta/h)y} + r_{33} h^4 y e^{(\beta/h)y} \\ & + r_{34} h^5 e^{(\beta/h)y} + r_{35} h^5). \end{aligned} \tag{31}$$

Substituting solutions (31) into equation (16) and eliminating h_t appearing in the first-order solution by equation (16) itself, yields the following non-linear kinematic equation of surface height:

$$h_t + A(h)h_x + \alpha \frac{\partial}{\partial x}[B(h)h_x + C(h)h_{xxx}] = 0 \quad (32)$$

where

$$A(h) = T_1 h^2$$

$$B(h) = [(T_3 - T_1 T_2)Pe + (T_6 - T_1 T_5)Re]h^6 - 2T_4 \cot\gamma h^3$$

$$C(h) = 2\alpha^2 T_4 \sigma N^{-1/3} Re^{-2/3} h^3$$

in which T_i, a_i, r_i are given in the Appendix.

STABILITY ANALYSIS

In the unperturbed state, the non-dimensional thickness h is equal to 1. Therefore, the non-dimensional film thickness for the perturbed state could be expanded in the following form:

$$h = 1 + \eta \quad (33)$$

where η is the perturbation of the thickness. Substituting equation (33) into equation (32) and keeping the terms up to third order in η lead to the evolution equation of η

$$L\eta = -(A'\eta + \tfrac{1}{2}A''\eta^2)\eta_x - \alpha \frac{\partial}{\partial x}[(B'\eta + \tfrac{1}{2}B''\eta^2)\eta_x + (C'\eta + \tfrac{1}{2}C''\eta^2)\eta_{xxx}] + O(\eta^4) \quad (34)$$

where

$$L = \frac{\partial}{\partial t} + A\frac{\partial}{\partial x} + \alpha\left[B\frac{\partial^2}{\partial x^2} + C\frac{\partial^4}{\partial x^4}\right]$$

and the values of A, B and their derivatives are evaluated at $h = 1$.

For the linear stability analysis, the non-linear part of equation (34) is neglected and the normal mode solution is assumed as

$$\eta = \Gamma \exp[\mathrm{i}(x - ct)] + \bar{\Gamma}\exp[-\mathrm{i}(x - ct)] \quad (35)$$

then, the complex wave celerity becomes

$$c = c_r + \mathrm{i}c_i = A + \mathrm{i}\alpha(B - C). \quad (36)$$

If $c_i > 0$, the film flow is linearly unstable. On the other hand, if $c_i < 0$, the film flow is linearly stable.

For the non-linear stability analysis, we use the method of multiple scales

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1}$$

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 \quad (37)$$

then equation (34) becomes

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)(\varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3) = -\varepsilon^2 N_2 - \varepsilon^3 N_3 \quad (38)$$

where

$$L_0 = L$$

$$L_1 = \frac{\partial}{\partial t_1} + A\frac{\partial}{\partial x_1} + \alpha\left[2B\frac{\partial}{\partial x}\frac{\partial}{\partial x_1} + 4C\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial x_1}\right]$$

$$L_2 = \frac{\partial}{\partial t} + \alpha\left[B\frac{\partial^2}{\partial x_1^2} + 6C\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial x_1^2}\right]$$

$$N_2 = A'\eta_1\eta_{1x} + \alpha[B'(\eta_1\eta_{1xx} + \eta_{1x}^2) + C'(\eta_1\eta_{1xxxx} + \eta_{1x}\eta_{1xxx})]$$

$$N_3 = A'(\eta_1\eta_{2x} + \eta_2\eta_{1x} + \eta_1\eta_{1x_1}) + \tfrac{1}{2}A''\eta_1^2\eta_{1x} + \alpha[B'(2\eta_{1x}\eta_{2x} + \eta_1\eta_{2xx} + \eta_2\eta_{1xx_1} + 2\eta_1\eta_{1x_1} + 2\eta_{1x1x_1}) + B''(\tfrac{1}{2}\eta_1^2\eta_{1xx} + \eta_1\eta_{1x}^2) + C'(\eta_1\eta_{2xxxx} + \eta_{1xxxx}\eta_2 + \eta_{1x}\eta_{2xxx} + 4\eta_1\eta_{1xxxx_1} + 3\eta_{1x}\eta_{1xxx_1} + \eta_{1xxx}\eta_{1x_1}) + C''(\tfrac{1}{2}\eta_1^2\eta_{1xxxx} + \eta_1\eta_{1x}\eta_{1xxx})].$$

Equation (38) is then solved order by order to get the solution for the $O(\varepsilon)$ equation $-L_0\eta_1 = 0$

$$\eta_1 = \Gamma(x_1, t_1, t_2)\exp[\mathrm{i}(x - c_r t)] + \text{C.C.} \quad (39)$$

Then the solution of η_2 and the secular condition for the $O(\varepsilon^3)$ equation are

$$\eta_2 = C_1\Gamma^2 \exp[2\mathrm{i}(x - c_r t)] + \text{C.C.} \quad (40)$$

$$\frac{\partial \Gamma}{\partial t_2} + D_1\frac{\partial^2 \Gamma}{\partial x_1^2} - \varepsilon^{-2}C_i\Gamma + E_1\Gamma^2\bar{\Gamma} = 0 \quad (41)$$

where

$$C_1 = C_{1r} + \mathrm{i}C_{1i} = \frac{1}{4\alpha(4C - B)}2\alpha(B' - C') - \mathrm{i}A'$$

$$D_1 = B - 6C$$

$$E_1 = E_{1r} + \mathrm{i}E_{1i}$$

$$E_{1r} = -A'C_{1i} + \alpha[\tfrac{1}{2}(C'' - B'') + (7C' - B')C_{1r}]$$

$$E_{1i} = A'C_{1r} + \tfrac{1}{2}A'' + \alpha[(7C' - B')C_{1i}].$$

We shall use equation (41) to investigate the weakly non-linear behaviour of film flow. Firstly for a filtered wave, there is no spatial modulation and this solution may be written as

$$\Gamma = \zeta \exp[-\mathrm{i}\Lambda t_2] \quad (42)$$

substituting expression (42) into equation (41), and neglecting the diffusion term, we obtain the following results:

$$\varepsilon\zeta = \left(\frac{c_i}{E_{1r}}\right)^{1/2}$$

$$\varepsilon^2\Lambda = E_{1i}\left(\frac{c_i}{E_{1r}}\right). \tag{43}$$

From the form of $\varepsilon\zeta$ one knows that in the linearly unstable region ($c_i > 0$) the condition of existence of the supercritical wave is $E_{1r} > 0$, and $2\varepsilon\zeta$ is the final amplitude. On the other hand, in the linearly stable region ($c_i < 0$) if $E_{1r} < 0$ then the film flow has the behaviour of subcritical instability and $2\varepsilon\zeta$ is the amplitude of threshold.

It is known that experimental work to control the wave motion at a highly specific mode is an extremely difficult task; the presence of side-band disturbance can hardly be avoided in laboratories or in practice. Studies for this disturbance were given by Eckhaus [17], Stuart and Diprima [18] and Keefe [19]. Just like the analysis of ref. [6], in this study, the stable condition of the supercritical wave under side-band disturbance is $D_1 < 0$.

RESULTS AND DISCUSSION

From equation (36), we have the expressions of linear wave speed and amplitude rate as

$$c_r = T_1$$

$$\alpha c_i = \alpha^2[(T_3 - T_1T_2)\,Re\,Pr + (T_6 - T_1T_5)Re - 2T_4\cot\gamma - 2\alpha^2T_4\sigma N^{-1/3}\,Re^{-2/3}]. \tag{44}$$

It is noted that the linear component of wave speed, c_r, obtained in this paper is identical with equation (24) of ref. [15]. When the film is heated from below ($\beta < 0$) this speed increases as β decreases. Also, when the film is cooled from below ($\beta > 0$), this speed decreases as β increases. It can be shown in Fig. 2 that as $b \to 0$, $c_r = 2$; while as $\beta \to \infty$, $c_r = 0$, and $\beta \to -\infty$, $c_r = 3$.

The condition of linear stability or instability is dependent on the sign of αc_i. The neutral stability

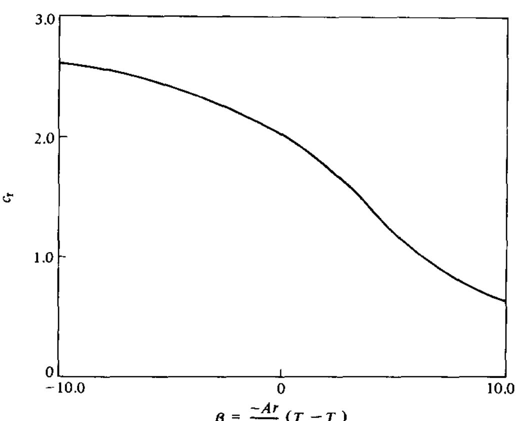

FIG. 2. Linear wave speed vs β.

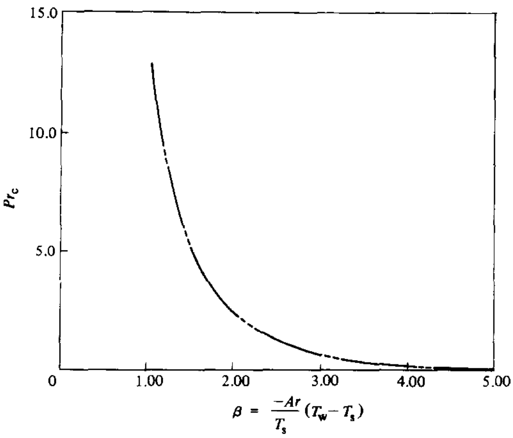

FIG. 3. Cut-off Prandtl number vs β.

curve ($\alpha c_i = 0$) which separates the α–Re plane into two regions; namely, the linearly unstable region ($\alpha c_i > 0$) where the small disturbance grows with time and the linearly stable region ($\alpha c_i < 0$) where the small disturbance decays with time. It is found that when the value of β is positive then the value of $(T_3 - T_1T_2)$ is negative, suggesting a cut-off Prandtl number, Pr_c, exists. When the Prandtl number is larger than this value the flow is linearly stable with respect to long waves. Figure 3 shows that the value of the cut-off Prandtl number decreases as the value of β increases.

From expression (44), it is found that the critical Reynolds number and the most rapidly growing linear mode are respectively

$$Re_c = \frac{2T_4\cot\gamma}{(T_3 - T_1T_2)\,Pr + (T_6 - T_1T_5)}$$

$$\alpha_n = \left\{\frac{N^{1/3}\,Re^{2/3}}{4T_4\sigma}[(T_3 - T_1T_2)Re\,Pr + (T_6 - T_1T_5)Re - 2T_4\cot\gamma]\right\}^{1/2}. \tag{45}$$

From Fig. 4 it is readily seen that increasing the

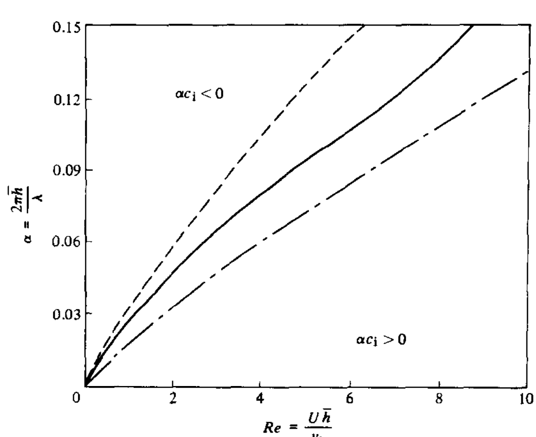

FIG. 4. Linear neutral curve with different values of β: $\sigma = 911.25$, $Pr = 7$, $\gamma = \pi/2$. ----, $\beta = -1$; ——, $\beta = 0.2$; — — —, $\beta = 1$.

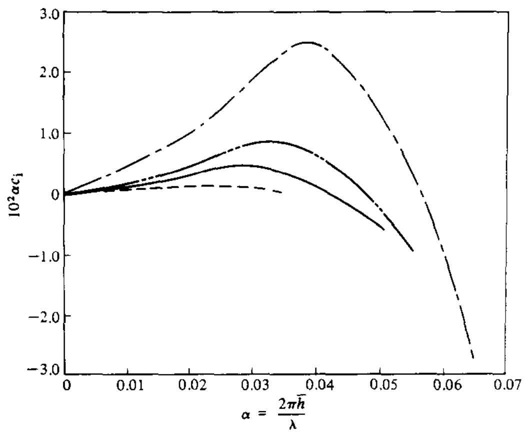

FIG. 5. Linear amplitude rate with different values of β: $Re = 2$, $Pr = 2$, $\sigma = 911.25$, $\gamma = \pi/2$. ———, $\beta = -1$; ———, $\beta = -0.2$; ———, $\beta = 0.2$; -----, $\beta = 1$.

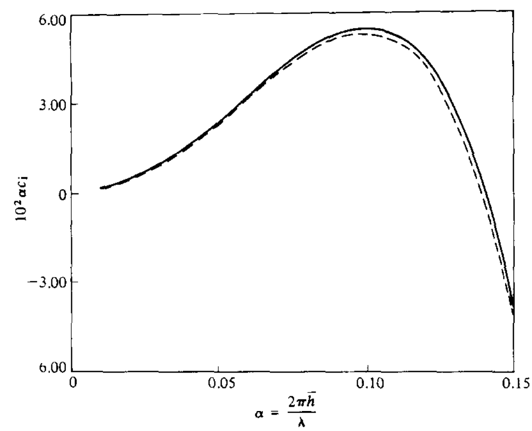

FIG. 7. Linear amplitude rate with different Prandtl numbers when $\beta < 0$, $\beta = -1.5$, $Re = 5$, $\gamma = \pi/2$, $\sigma = 911.25$. ------, $Pr = 2$; ———, $Pr = 4$.

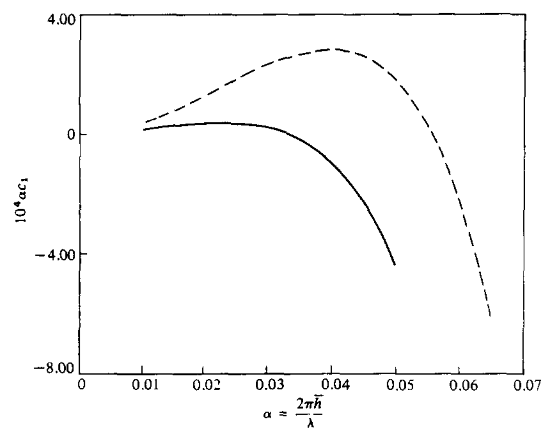

FIG. 6. Linear amplitude rate with different Prandtl numbers when $\beta > 0$, $\beta = 1.5$, $Re = 5$, $\gamma = \pi/2$. ------, $Pr = 2$; ———, $Pr = 4$.

value of β will increase the linearly stable region in the α–Re plane. Also, Fig. 5 shows that decreasing the value of β will increase the linear amplitude rate. The above results both indicate that cooling stabilizes the film flow, while heating destabilizes it.

Figure 6 shows that increasing the Prandtl number will decrease the linear amplitude rate when $\beta > 0$. Figure 7 shows that increasing the Prandtl number will slightly increase the linear amplitude rate when $\beta < 0$.

The non-linear stability analysis is used to study whether the finite amplitude disturbance in the linearly stable region will cause instability (subcritical instability), as well as to study whether the subsequent non-linear evolution of disturbance in the linearly unstable region will develop into new equilibrium with finite amplitude (supercritical stability) or such evolution will grow towards an explosive state. A review of the characteristics of equation (40) will show that the negative value of E_{1r} tends to cause the system to be nonlinearly unstable. Such instability in the linearly stable region is called subcritical instability. That is, when the amplitude of disturbance is larger than that of the threshold, then the amplitude will grow although the prediction from the linear theory is stable. On the other hand, such instability in the linearly unstable region will lead the system to an explosive state which could be considered as solutions of complex patterns.

It is observed, the shaded regions of Figs. 8 and 9, that both subcritical instability ($c_i < 0$, $E_{1r} < 0$) and the explosive solution ($c_i > 0$, $E_{1r} < 0$) are possible for the film flow. Additionally, it is found that increasing the value of β will decrease the areas of regions of non-linear instability in the α–Re plane. It is also observed that supercritical stability ($c_i > 0$, $E_{1r} < 0$, blank region in the linearly unstable region) is possible near the region of the upper branch of the linearly

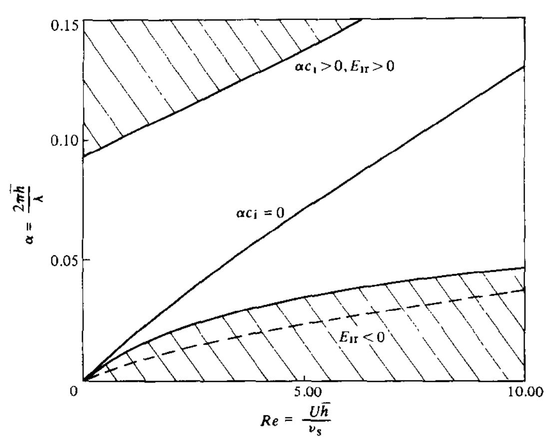

FIG. 8. Non-linear stability curve for the case of cooling: $\beta = 1$, $\sigma = 911.25$, $Pr = 7$, $\gamma = \pi/2$. -------, Side-band neutral stability curve.

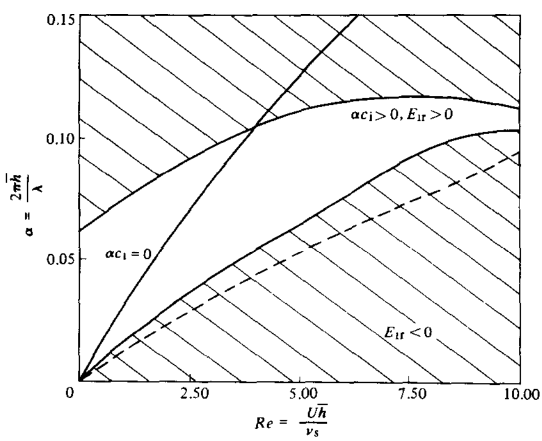

FIG. 9. Non-linear stability curve for the case of heating: $\beta = -1$, $\sigma = 911.25$, $Pr = 7$, $\gamma = \pi/2$. ------, Side-band neutral stability curve.

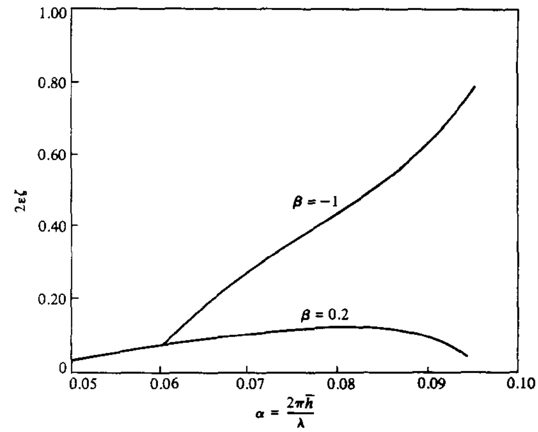

FIG. 11. Finite amplitude of supercritical wave with different values of β: $Re = 5$, $\sigma = 911.25$, $Pr = 7$, $\gamma = \pi/2$.

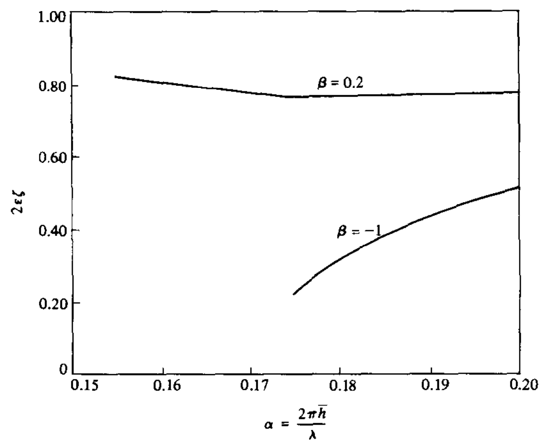

FIG. 10. The amplitude of threshold in the subcritical unstable region: $Re = 7$, $\sigma = 911.25$, $Pr = 7$, $\gamma = \pi/2$.

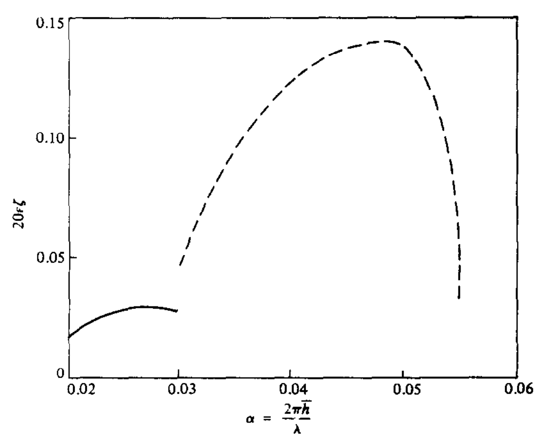

FIG. 12. Finite amplitude of supercritical wave with different Prandtl numbers when $\beta > 0$, $\beta = 1.5$, $Re = 5$, $\sigma = 911.25$, $\gamma = \pi/2$. ------, $Pr = 2$; ———, $Pr = 4$.

neutral curve. In such a case, filtered waves are linearly stable when subject to side-band disturbance.

It is interesting to note that ref. [4] predicted the possibility of the existence of subcritical instability for the film flow, but it did not discover the explosive solution in the region near the lower branch of the neutral stability curve. In contradiction to this, refs. [5, 6, 8] indicated that the subcritical instability was not possible for the film flow. In reality, the possibility of existence of subcritical instability and supercritical stability was pointed out by some researchers [20, 21]. From our viewpoint, especially for the case of heating from the wall ($\beta < 0$), those results of previous studies of refs. [4–8] perhaps have expressed some aspects of the film flow system but the description was not adequate.

Figure 10 displays the amplitude of threshold in the subcritical unstable region with different values of β. It is found that heating will decrease such an amplitude. From Fig. 11, we find that the finite amplitude of a supercritical wave will increase as β decreases.

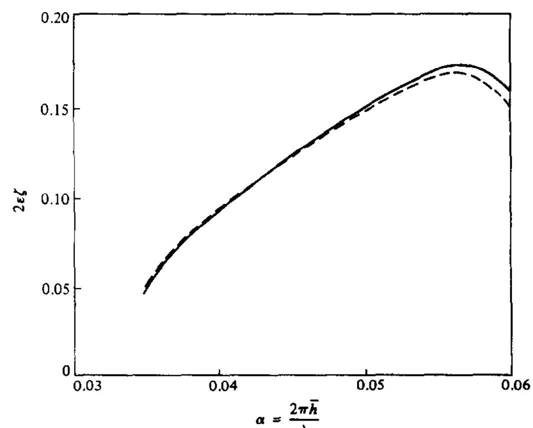

FIG. 13. Finite amplitude of supercritical wave with different Prandtl numbers when $\beta < 0$, $\beta = -1.5$, $Re = 2$, $\sigma = 911.25$, $\gamma = \pi/2$. ------, $Pr = 2$; ———, $Pr = 4$.

Figure 12 shows that increasing the Prandtl number

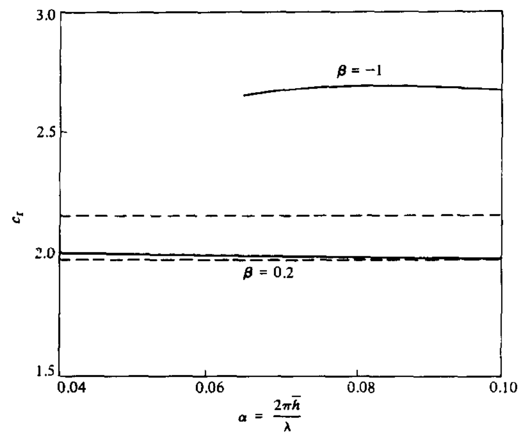

FIG. 14. Linear and non-linear wave speed with different values of β: $Re = 5$, $\sigma = 911.25$, $Pr = 7$, $\gamma = \pi/2$. ----, c_r; ———, $c_r + \varepsilon^2\Lambda$.

will decrease the amplitude of supercritical waves when $\beta > 0$. Figure 13 shows that increasing the Prandtl number will slightly increase the amplitude of the supercritical wave when $\beta < 0$. Also, Fig. 14 shows that the difference of non-linear wave speed ($c_r + \varepsilon^2\Lambda$) and linear wave speed (c_r) increases when the value of β decreases.

It is clear that, from the choice of reference temperature in this study, cooling results in a virtually more viscous fluid, while heating results in a less viscous one. This is the reason why cooling causes film flow more stable than heating does.

The above theory has the following three limitations for applications.

(1) Refer to equations (15) and (24), when Reynolds number becomes zero, a singular point exists. The results might not be applied at this point and need further modifications.

(2) The value of α cannot be too large, since long waves are addressed in this analysis. In the case of cooling, the value of β can never exceed the cut-off value as pointed out by ref. [16].

(3) In this study $\alpha^2\sigma$ is taken to be of $O(1)$ so that the applications may be invalid in the region $O(\alpha) \ll O(\sigma^{-1/2})$. For most of the known liquids the low bound of $O(\alpha)$ is about of $O(10^{-2})$ here.

CONCLUSION

In this study, a non-linear kinematic equation for film thickness taking into account the effect of viscosity variation is used to investigate both the linear and non-linear stabilities of film flows. Since the viscosity variation is mainly caused by the thermal effect and the interfacial temperature is taken as the reference temperature, hence, cooling from the wall results in a more viscous fluid, while heating results in a less viscous one. The parameter, β, the gradient of viscosity, is introduced here and its value is positive (negative) when cooling (heating) from the wall.

For the linear theory, a closed form solution is reached, from which the critical Reynolds number and most unstable linear mode are obtained analytically. It is found that the linear waves are travelling at three times the speed of the unperturbed surface as β approaches $-\infty$; as β approaches ∞, the linear wave speed approaches zero. It is also found that the linear amplitude rate increases as the value of β decreases. For the case of cooling, a cut-off Prandtl number exists. For values above this number, the flow is stable with respect to long-wave disturbance. Increasing the Prandtl number will stabilize (destabilize) the film flow when $\beta > 0$ (<0).

The non-linear stability analysis shows that, especially in the case of heating from the wall ($\beta < 0$), both supercritical stability and subcritical instability are possible for the film flow system. The nonlinearly unstable region in the α–Re (wave number–Reynolds number) plane will increase when the value of β decreases. Also, decreasing the value of β will reduce the amplitude of threshold in the subcritical unstable region and will increase the amplitude of the supercritical wave.

To wrap up, heating from the wall will linearly and non-linearly destabilize the film flow system, while cooling from the wall will yield exactly the reverse results.

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APPENDIX

$$l_1 = 1/(e^\beta - \beta - 1)$$
$$a_1 = -\beta$$
$$a_2 = \beta + 1$$
$$a_3 = -\beta$$
$$a_4 = \beta + 4$$
$$a_5 = -3(1 + 2/\beta)$$
$$a_6 = 2(\beta + 1)$$
$$b_1 = a_1 - a_3$$
$$b_2 = a_2 - a_4$$
$$b_3 = -a_5$$
$$b_4 = -a_2 - a_6$$
$$d_1 = b_1/\beta^2$$
$$d_2 = (-4b_1 b_2 \beta)/\beta^3$$
$$d_3 = (6b_1 - 2b_2\beta + b_3\beta^2)/\beta^4$$
$$d_4 = b_4/b_1$$
$$d_5 = -b_3/2$$
$$d_6 = -(d_1 + d_2 + d_3)\,e^\beta - (d_4 + d_5 - d_3)$$
$$d_7 = l_1 a_1 \beta^2 e^\beta$$
$$d_8 = (a_1 + a_2\beta) l_1 \beta\, e^\beta$$
$$r_1 = \beta^2 l_1$$
$$r_2 = -\beta(\beta + 4) l_1$$
$$r_3 = (-\beta^2 + 3\beta + 12) l_1$$
$$r_4 = \frac{1}{\beta}(\beta^3 + 2\beta^2 - 6\beta - 24) l_1$$
$$r_5 = \frac{1}{\beta^2}(-\beta^3 - 2\beta^2 + 6\beta + 24) l_1$$
$$r_6 = \frac{1}{\beta^2}(\beta^3 + 2\beta^2 - 6\beta - 24) l_1$$
$$r_7 = 0$$
$$r_8 = -\tfrac{1}{2} l_1$$
$$r_9 = \frac{3}{2\beta}(2\beta + 3) l_1$$
$$r_{10} = \frac{1}{4\beta^2}(-6\beta^2 - 6\beta + 31) l_1$$
$$r_{11} = \tfrac{1}{2}\beta^2(\beta + 1) l_1$$
$$r_{12} = (\tfrac{1}{2}\beta^3 + \beta^2 - \beta) l_1$$
$$r_{13} = \frac{1}{\beta}(-\tfrac{1}{2}\beta^4 - 3\beta^2 - 2\beta - 12 - 12\,e^\beta) l_1$$
$$r_{14} = \frac{1}{\beta^2}(-\tfrac{1}{2}\beta^5 + \tfrac{1}{2}\beta^4 + \beta^3 + \tfrac{3}{2}\beta^2 - 14\beta + 24 + 24\,e^\beta + 12\,e^\beta) l_1$$
$$r_{15} = \frac{1}{\beta^3}(\beta^6 + 3\beta^5 + \tfrac{11}{2}\beta^4 + \tfrac{13}{2}\beta^3 + \tfrac{57}{2}\beta^2 + 14\beta$$
$$-24 - 2\beta^3 e^\beta - 8\beta^2 e^\beta - 32\beta\, e^\beta - 24\,e^\beta) l_1$$
$$r_{16} = \frac{1}{\beta^3}(-\beta^6 - 3\beta^5 - 11\beta^4 - 5\beta^3 - 27\beta^2 - \tfrac{87}{4}\beta$$
$$+24 + 2\beta^3 e^\beta + 8\beta^2 e^\beta + 32\beta\, e^\beta + 24\,e^\beta) l_1$$
$$r_{17} = \frac{1}{\beta} e^{-\beta}$$
$$r_{18} = -\frac{1}{\beta^2}(\beta + 1)\, e^{-\beta}$$
$$r_{19} = \frac{1}{\beta^2}(\beta + 1)\, e^{-\beta}$$
$$r_{20} = \tfrac{1}{2} e^{-\beta}$$
$$r_{21} = \frac{-1}{2\beta}(\beta + 5)\, e^{-\beta}$$
$$r_{22} = \frac{1}{\beta^2}(\beta + 3)\, e^{-\beta}$$
$$r_{23} = -\frac{2}{\beta}(\beta + 1)\, e^{-\beta}$$
$$r_{24} = \frac{2}{\beta^2}(\beta^2 + 4\beta + 2)\, e^{-\beta}$$
$$r_{25} = \frac{-1}{4\beta^2}(8\beta^2 + 33\beta + 19)\, e^{-\beta}$$
$$r_{26} = 0$$
$$r_{27} = -\tfrac{1}{6} e^{-\beta}$$
$$r_{28} = \frac{1}{9\beta^3}(\beta^3 + \tfrac{11}{2}\beta^2)\, e^{-\beta}$$
$$r_{29} = \frac{-1}{27\beta^4}(\tfrac{9}{2}\beta^4 + \tfrac{29}{2}\beta^3 + \tfrac{13}{4}\beta^2)\, e^\beta$$
$$r_{30} = -\tfrac{3}{2}(\beta + 1)\, e^{-\beta}$$
$$r_{31} = \frac{1}{2\beta}(3\beta^2 + 13\beta + 7)\, e^{-\beta}$$
$$r_{32} = -\frac{1}{4\beta^2}(11\beta^2 + 37\beta + 23)\, e^{-\beta}$$
$$r_{33} = \frac{2}{\beta}(\beta + 1)^2\, e^{-\beta}$$
$$r_{34} = \frac{1}{\beta^2}[-5\beta^2 + 4\beta + 6 - \tfrac{1}{4}e^\beta - 2\beta(\beta + 1)^2 e^{-\beta}]\, e^{-\beta}$$
$$r_{35} = \frac{1}{108\beta^2}[855\beta^2 + 625\beta - 14 + 27\,e^\beta$$
$$+216\beta(\beta + 1)^2 e^{-\beta}]\, e^{-\beta}$$

$$T_1 = \frac{6}{\beta}\left[\frac{e^{\beta}-\beta^2/2-\beta-1}{e^{\beta}-\beta-1}\right]$$

$$T_2 = \frac{1}{6}\left\{\frac{e^{\beta}}{\beta^5}r_1(\beta^4-4\beta^3+12\beta^2-24\beta+24)\right.$$
$$+r_2(\beta^4-3\beta^3+6\beta^2-6\beta)+r_3(\beta^4-2\beta^3+2\beta^2)$$
$$+r_4(\beta^4-\beta^3)+r_5\beta^4+r_6$$
$$\left.-\frac{1}{\beta^5}(24r_1-6r_2\beta+2r_3\beta^2-r_4\beta^3+r_5\beta^4)\right\}$$

$$T_3 = l_1\left\{\frac{e^{2\beta}}{8\beta^4}[r_7(4\beta^3-6\beta^2+6\beta-3)\right.$$
$$+r_8(4\beta^3-4\beta^2+2\beta)+r_9(4\beta^3-2\beta^2)$$
$$+4r_{10}\beta^3]+\frac{e^{\beta}}{\beta^5}[r_{11}(\beta^4-4\beta^3+12\beta^2-24\beta+24)$$
$$+r_{12}(\beta^4-3\beta^3+6\beta^2-6\beta)+r_{13}(\beta^4-2\beta^3+2\beta^2)$$
$$+r_{14}(\beta^4-\beta^3)+r_{15}\beta^4]+r_{16}$$
$$-\frac{1}{8\beta^4}(-3r_7+2r_8\beta-2r_9\beta^2+4r_{10}\beta^3)$$
$$\left.-\frac{1}{\beta^5}(24r_{11}-6r_{12}\beta+2r_{13}\beta^2-r_{14}\beta^3+r_{15}\beta^4)\right\}$$

$$T_4 = -[(r_{17}\beta-r_{17}+r_{18}\beta)e^{\beta}+r_{19}\beta^2-r_{18}\beta+r_{17}]/\beta^2$$

$$T_5 = l_1\left\{\frac{e^{2\beta}}{4\beta^3}[r_{20}(2\beta^2-2\beta+1)+r_{21}(2\beta^2-\beta)\right.$$
$$+2r_{22}\beta^2]+\frac{e^{\beta}}{\beta^2}[r_{23}(\beta-1)+r_{24}\beta]+r_{25}$$
$$\left.-\frac{1}{4\beta^3}(r_{20}-r_{21}\beta+2r_{22}\beta^2)-\frac{1}{\beta^2}(-r_{23}+r_{24}\beta)\right\}$$

$$T_6 = N^2\left\{\frac{e^{3\beta}}{27\beta^4}[r_{26}(9\beta^3-9\beta^2+6\beta-2)\right.$$
$$+r_{27}(9\beta^3-6\beta^2+2\beta)+r_{28}(9\beta^3-3\beta^2)+9r_{29}\beta^3]$$
$$+\frac{e^{2\beta}}{4\beta^3}[r_{30}(2\beta^2-2\beta+1)+r_{31}(2\beta^2-\beta)$$
$$+2r_{32}\beta^2]+\frac{e^{\beta}}{\beta^2}[r_{33}(\beta-1)+r_{34}\beta]+r_{35}$$
$$-\frac{1}{4\beta^3}(r_{30}-r_{31}\beta+2r_{32}\beta^2)-\frac{r_{33}}{\beta}$$
$$\left.-\frac{1}{27\beta^4}(-2r_{26}+2r_{27}\beta-3r_{28}\beta^2+9r_{29}\beta^2)\right\}.$$

ANALYSE DE STABILITE NON LINEAIRE D'UN FILM TOMBANT SUR UN PLAN INCLINE CHAUFFE OU REFROIDI, AVEC VARIATION DE VISCOSITE

Résumé—Les équations cinématiques non linéaires pour l'épaisseur du film, prenant en compte l'effet de variation de la viscosité selon le type Arrhenius sont utilisées pour étudier la stabilité non linéaire des écoulements en film. Les résultats montrent que le refroidissement (chauffage) par la paroi stabilise (déstabilise) les écoulements en film, à la fois linéairement et non linéairement. La stabilité supercritique et l'instabilité sous-critique prouvent possible, avec un chauffage plus intense, la réduction de l'amplitude de seuil dans la région instable sous-critique et l'accroissement de l'amplitude des ondes supercritiques. La stabilité est aussi influencée par le nombre de Prandtl de telle manière que la stabilité est accrue (diminuée) quand la valeur du nombre de Prandtl augmente, avec le refroidissement (chauffage).

UNTERSUCHUNG DER STABILITÄT VON FILMSTRÖMUNGEN ENTLANG EINER BEHEIZTEN ODER GEKÜHLTEN, GENEIGTEN PLATTE MIT HILFE VON NICHTLINEAREN BEZIEHUNGEN UND VARIATION DER VISKOSITÄT

Zusammenfassung—Bei dieser Untersuchung über die Stabilität von Filmströmungen werden nichtlineare Bewegungsgleichungen für die Filmdicke benutzt, die eine Variation der Viskosität nach der Arrhenius-Beziehung berücksichtigen. Die Ergebnisse zeigen, daß das Kühlen (Beheizen) der Wand die Filmströmung sowohl bei linearer als auch bei nichtlinearer Betrachtung stabilisiert (destabilisiert). Es ist möglich, sowohl oberhalb der Grenzbedingung Stabilität als auch im unterkritischen Bereich Instabilität zu erhalten, wobei eine stärkere Beheizung im instabilen Gebiet den Grenzwert der Amplitude erniedrigt bzw. die Amplitude von überkritischen Wellen vergrößert. Außerdem beeinflußt die Prandtl-Zahl die Stabilität: bei steigender Prandtl-Zahl und Kühlung (Beheizung) der Platte wird der Stabilitätsbereich erweitert (verringert).

АНАЛИЗ НЕЛИНЕЙНОЙ УСТОЙЧИВОСТИ ДВИЖЕНИЯ ПЛЕНКИ ЖИДКОСТИ С ПЕРЕМЕННОЙ ВЯЗКОСТЬЮ, СТЕКАЮЩЕЙ ПО НАГРЕТОЙ ИЛИ ОХЛАЖДЕННОЙ ПЛОСКОСТИ

Аннотация—Для исследования нелинейной устойчивости течения пленки жидкости с вязкостью, изменяющейся по соотношению типа Аррениуса, используются нелинейные кинематические уравнения для толщины пленки. Результаты показывают, что охлаждение (нагрев) со стороны стенки будет стабилизировать (дестабилизировать) течение пленки жидкости как в линейном, так и нелинейном приближении. В случае сильного нагрева подтверждено наличие сверхкритической устойчивости и докритической неустойчивости, приводящих к уменьшению пороговой амплитуды в докритической неустойчивой зоне и увеличению амплитуды сверхкритических волн. На устойчивость также оказывает влияние число Прандтля: с его ростом устойчивость течения при охлаждении (нагреве) увеличивается (уменьшается).